

# Highest coefficient of scalar products in $SU(3)$ -invariant integrable models

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## Abstract

We study  $SU(3)$ -invariant integrable models solvable by nested algebraic Bethe ansatz. Scalar products of Bethe vectors in such models can be expressed in terms of a bilinear combination of their highest coefficients. We obtain various different representations for the highest coefficient in terms of sums over partitions. We also obtain multiple integral representations for the highest coefficient.

## 1 Introduction

The problem of calculating local operators form factors and correlation functions in quantum integrable models is of highest importance. When integrable models are solvable by algebraic Bethe ansatz [1, 2, 3] this problem can be reduced to the calculation of scalar products of Bethe vectors.

The scalar products of Bethe vectors were first considered for  $\mathfrak{gl}_2$ -based integrable models [4, 5]. In these works the notion of highest coefficient  $K_n$  of a scalar product was introduced.

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Any scalar product can be expressed in terms of a bilinear combination of  $K_n$  (Izergin–Korepin formula). It was shown in [4, 5] that for the models with  $SU(2)$ -symmetry (and  $q$ -deformed  $SU(2)$ -symmetry)  $K_n$  is equal to the partition function of the six-vertex model with domain wall boundary conditions. An explicit determinant representation for this partition function was derived in [6].

A wide class of quantum integrable models is associated with higher rank algebras  $\mathfrak{gl}_N$ . An algebraic Bethe ansatz for these type of models is called hierarchical (or nested) and was introduced in [7] (see also [8]). The first results concerning the scalar products in the models with  $SU(3)$ -invariant  $R$ -matrix was obtained by N.Yu. Reshetikhin in [9]. There, an analog of Izergin–Korepin formula for the scalar product of generic Bethe vectors and a determinant representation for the norm of the transfer-matrix eigenvectors were found. Similarly to the Izergin–Korepin formula Reshetikhin’s representation for the scalar product can be considered as a bilinear combination of highest coefficients ( $Z_{a,b}$ ). In turn,  $Z_{a,b}$  is equal to a special partition function. The study of this partition function is the subject of the present paper.

Recently the explicit representation for the  $Z_{a,b}$  associated with  $SU(3)$ -invariant  $R$ -matrix was obtained in [10]. There,  $Z_{a,b}$  was given as a trilinear combination of  $K_n$ . There exist, however, many other representations of similar type. We have found it very useful to use different representations for the  $Z_{a,b}$  in studying the problem of scalar products. In particular, this approach allowed us to derive a determinant representation for the scalar product of eigenvectors of the transfer-matrix and twisted transfer-matrix (see our forthcoming publication [11]). In the present paper we prove several representations for  $Z_{a,b}$  in terms of sums over partitions and in terms of multiple integrals of Cauchy type.

The article is organized as follows. In Section 2 we give the definition of the partition function equivalent to  $Z_{a,b}$  and explain the notations used below. Section 3 gathers our results: first, we give a list of sum formulas for the highest coefficient  $Z_{a,b}$  (section 3.1), then we provide integral representations for  $Z_{a,b}$  (section 3.2), and finally, we show recursion relations on the highest coefficient  $Z_{a,b}$ , that allow one to fix it unambiguously (section 3.3). The following sections deal with the proofs of our results: in Sections 4 and 5 we prove the different representations given in Section 3, and in Section 6 we prove the recursion relations. Some properties of the highest coefficient  $K_n$ , needed for our calculations, are given in Appendix A. In Appendix B we prove the absence of contribution of certain poles in the integral representations for  $Z_{a,b}$ .

## 2 Definitions and notations

The  $SU(3)$ -invariant  $R$ -matrix has the form

$$R(x, y) = \mathbf{I} + g(x, y)\mathbf{P}, \quad g(x, y) = \frac{c}{x - y}, \quad (2.1)$$

where  $\mathbf{I}$  is the identity matrix,  $\mathbf{P}$  is the permutation matrix,  $c$  is a constant. Keeping in mind possible generalization of our results to the models with  $q$ -deformed  $SU(3)$ -symmetry we do not stress that the function  $g(x, y)$  depends on the difference  $x - y$ .

Apart from the function  $g(x, y)$  we also introduce a function  $f(x, y)$  as

$$f(x, y) = \frac{x - y + c}{x - y}. \quad (2.2)$$

Clearly in our case  $f(x, y) = 1 + g(x, y)$ , however it is no more true in the  $q$ -deformed case. Two other auxiliary functions will be also used

$$h(x, y) = \frac{f(x, y)}{g(x, y)} = \frac{x - y + c}{c}, \quad t(x, y) = \frac{g(x, y)}{h(x, y)} = \frac{c^2}{(x - y)(x - y + c)}. \quad (2.3)$$

The following obvious properties of the functions introduced above are useful

$$g(x, y) = -g(y, x), \quad h(x - c, y) = g^{-1}(x, y), \quad f(x - c, y) = f^{-1}(y, x), \quad t(x - c, y) = t(y, x). \quad (2.4)$$

The  $R$ -matrix (2.1) satisfies Yang–Baxter equation

$$R_{12}(x, y)R_{13}(x, z)R_{23}(y, z) = R_{23}(y, z)R_{13}(x, z)R_{12}(x, y). \quad (2.5)$$

The equation (2.5) holds in the tensor product  $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ . The subscripts of the  $R$ -matrices in (2.5) show the spaces where the given  $R$ -matrix acts non-trivially.

In order to define the partition function, which is equivalent to  $Z_{a,b}$  we use graphical representation of the  $R$ -matrix (see [9] for details). We picture the  $R(x, y)$  by a vertex, in which the horizontal and vertical lines are associated with the spectral parameters  $x$  and  $y$  respectively (see Fig. 1). The edges of the vertex are labeled by the matrix indices of  $(R)_{jk, \ell m}$ .

We also consider the  $R$ -matrix  $R^{t_1}(y, x)$ , where  $t_1$  means the transposition with respect to the first space. This  $R$ -matrix is denoted by a dotted vertex (see Fig. 1).

$$\begin{array}{c} y \\ | \\ x \overset{j}{\text{---}} \overset{k}{\text{---}} \\ | \\ \ell \end{array} = (R(x, y))_{jk, \ell m} \quad \begin{array}{c} y \\ | \\ x \overset{j}{\text{---}} \overset{k}{\text{---}} \\ | \\ \ell \end{array} = (R^{t_1}(y, x))_{jk, \ell m}$$

Figure 1: Graphical pictures of  $R(x, y)$  and  $R^{t_1}(y, x)$ .

Due to (2.1) there exists three types of vertices corresponding to non-zero entries of  $R(x, y)$  or  $R^{t_1}(y, x)$ . Following Baxter's terminology we call these vertices  $a$ -type,  $b$ -type, and  $c$ -type [12]. The  $a$ -type vertex has all four indices equal to each other:  $j = k = \ell = m$ . The corresponding statistical weights are equal to  $f(x, y)$  for usual vertex and  $f(y, x)$  for dotted vertex (see Fig. 2, Fig. 3). For the  $b$ -type vertex, we have  $j = k, \ell = m, j \neq \ell$  and statistical weights are equal to 1. Finally  $j = \ell, k = m, j \neq k$  for the  $c$ -type vertex and  $j = m, k = \ell, j \neq k$  for the  $c$ -type dotted vertex. The statistical weights are  $g(x, y)$  and  $g(y, x)$  respectively.

$$\begin{array}{ccc} \begin{array}{c} y \\ | \\ x \overset{j}{\text{---}} \overset{j}{\text{---}} \\ | \\ j \end{array} = f(x, y) & \begin{array}{c} y \\ | \\ x \overset{k}{\text{---}} \overset{k}{\text{---}} \\ | \\ j \end{array} = 1 & \begin{array}{c} y \\ | \\ x \overset{j}{\text{---}} \overset{k}{\text{---}} \\ | \\ j \end{array} = g(x, y) \\ a\text{-type vertex} & b\text{-type vertex} & c\text{-type vertex} \end{array}$$

Figure 2:  $a, b, c$  vertices and their statistical weights.

Before giving the definition of the highest coefficient  $Z_{a,b}$ , we describe the notations used below. We always denote sets of variables by bar:  $\bar{x}, \bar{y}, \bar{w}$  etc. Individual elements of the sets



The partition function introduced by Reshetikhin depends on four sets of variables. We denote it by  $Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y})$ . The subscripts show that  $\#\bar{t} = \#\bar{x} = a$  and  $\#\bar{s} = \#\bar{y} = b$ . We separate the sets with the same number of elements by semicolon in order to stress that  $Z_{a,b}$  is not symmetric with respect to the changing of their order, for instance,  $Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) \neq Z_{a,b}(\bar{x}; \bar{t} | \bar{s}; \bar{y})$ . The graphical representation of the function  $Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y})$  is shown on Fig. 4.

$$Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) =$$

As usual

where  $L(\text{vertex})$  is the statistical weight corresponding to given vertex.

To conclude this section we introduce one more convention concerning the notations. In order to avoid too cumbersome formulas below we use shorthand notations for the products of functions  $g(x, y)$ ,  $f(x, y)$ ,  $h(x, y)$ , and  $t(x, y)$ . Originally these functions depend on two variables. We use the notations  $g(\bar{x}, \bar{y})$ ,  $f(t_k, \bar{y})$  etc. for the products of these functions with

respect to the corresponding sets. For example,

$$\begin{aligned} h(\bar{y}, \bar{s}) &= \prod_{y_j \in \bar{y}} \prod_{s_k \in \bar{s}} h(y_j, s_k); & g(x_k, \bar{w}) &= \prod_{w_j \in \bar{w}} g(x_k, w_j); \\ f(\bar{x}_{\bar{p}}, x_p) &= \prod_{x_j \in \bar{x} \setminus x_p} f(x_j, x_p); & f(\bar{s}_{\Pi}, \bar{s}_I) &= \prod_{s_j \in \bar{s}_{\Pi}} \prod_{s_k \in \bar{s}_I} f(s_j, s_k). \end{aligned} \quad (2.7)$$

### 3 Main results

As we have mentioned already there exists several different representations for  $Z_{a,b}$ . At this time, it is not clear to us, which one will be the most convenient for further work. Therefore we give a whole list of different representations: hopefully the right one will be among them.

First of all we recall the determinant formula for  $K_n$  (or, what is the same, for the partition function of the six-vertex model with domain wall boundary conditions) [6]. We denote it by  $K_n(\bar{x}|\bar{y})$ . The subscript  $n$  means that  $\#\bar{x} = \#\bar{y} = n$ .  $K_n$  is given by

$$K_n(\bar{x}|\bar{y}) = \Delta'_n(\bar{x}) \Delta_n(\bar{y}) h(\bar{x}, \bar{y}) \det_n t(x_j, y_k), \quad (3.1)$$

where

$$\Delta_n(\bar{y}) = \prod_{j < k}^n g(y_j, y_k), \quad \Delta'_n(\bar{x}) = \prod_{j > k}^n g(x_j, x_k). \quad (3.2)$$

All representations for  $Z_{a,b}$  involve  $K_n$ .

#### 3.1 Sum formulas

We first give several formulas for  $Z_{a,b}$  in terms of sums over partitions of certain sets. In all the representations given below two sets of arguments are fixed, while the two other sets are divided into subsets.

- *The sum over partitions of  $\bar{s}$  and  $\bar{x}$ .*

$$Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = (-1)^b \sum K_b(\bar{s} - c|\bar{w}_I) K_a(\bar{w}_{\Pi}|\bar{t}) K_b(\bar{y}|\bar{w}_I) f(\bar{w}_I, \bar{w}_{\Pi}). \quad (3.3)$$

Here  $\bar{w} = \{\bar{s}, \bar{x}\}$ . The sum is taken with respect to partitions of the set  $\bar{w}$  into subsets  $\bar{w}_I$  and  $\bar{w}_{\Pi}$  with  $\#\bar{w}_I = b$  and  $\#\bar{w}_{\Pi} = a$ .

There exists slightly different representation, so-called twin formula:

$$Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = (-1)^a \sum K_a(\bar{w}_{\Pi} - c|\bar{x}) K_a(\bar{w}_{\Pi}|\bar{t}) K_b(\bar{y}|\bar{w}_I) f(\bar{w}_I, \bar{w}_{\Pi}). \quad (3.4)$$

All the notations are the same as in (3.3). If we set explicitly  $w_I = \{\bar{s}_I, \bar{x}_{\Pi}\}$  and  $\bar{w}_{\Pi} = \{\bar{s}_{\Pi}, \bar{x}_I\}$  with  $\#\bar{s}_{\Pi} = \#\bar{x}_{\Pi} = k$ , then the equivalence of (3.3) and (3.4) becomes evident. Indeed, we have due to (A.4)

$$\begin{aligned} (-1)^a K_a(\bar{w}_{\Pi} - c|\bar{x}) &= (-1)^a K_a(\bar{s}_{\Pi} - c, \bar{x}_I - c|\bar{x}) = (-1)^k K_k(\bar{s}_{\Pi} - c|\bar{x}_{\Pi}) \\ &= (-1)^b K_b(\bar{s} - c|\bar{s}_I, \bar{x}_{\Pi}) = (-1)^b K_b(\bar{s} - c|\bar{w}_I). \end{aligned} \quad (3.5)$$

The representation (3.3) with specification  $w_I = \{\bar{s}_I, \bar{x}_I\}$  and  $\bar{w}_{II} = \{\bar{s}_{II}, \bar{x}_{II}\}$  was proved in [10].

- The sum over partitions of  $\bar{y}$  and  $\bar{t}$ .

$$Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = (-1)^a f(\bar{y}, \bar{x}) f(\bar{s}, \bar{t}) \sum K_a(\bar{t} - c|\bar{\eta}_I) K_a(\bar{x}|\bar{\eta}_I) K_b(\bar{\eta}_{II} - c|\bar{s}) f(\bar{\eta}_I, \bar{\eta}_{II}). \quad (3.6)$$

Here  $\bar{\eta} = \{\bar{y} + c, \bar{t}\}$ . The sum is taken with respect to partitions of the set  $\bar{\eta}$  into subsets  $\bar{\eta}_I$  and  $\bar{\eta}_{II}$  with  $\#\bar{\eta}_I = a$  and  $\#\bar{\eta}_{II} = b$ . This formula also has a twin

$$Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = (-1)^b f(\bar{y}, \bar{x}) f(\bar{s}, \bar{t}) \sum K_b(\bar{\eta}_{II} - c|\bar{y} + c) K_a(\bar{x}|\bar{\eta}_I) K_b(\bar{\eta}_{II} - c|\bar{s}) f(\bar{\eta}_I, \bar{\eta}_{II}). \quad (3.7)$$

- The sum over partitions of  $\bar{t}$  and  $\bar{x}$ .

$$Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = \sum (-1)^n f(\bar{s}, \bar{t}_I) f(\bar{y}, \bar{x}_{II}) f(\bar{t}_I, \bar{t}_{II}) f(\bar{x}_{II}, \bar{x}_I) \\ \times K_n(\bar{x}_I|\bar{t}_I) K_{a-n}(\bar{x}_{II}|\bar{t}_{II} - c) K_{b+n}(\bar{y}, \bar{t}_I - c|\bar{s}, \bar{x}_I). \quad (3.8)$$

The sum is taken with respect to all partitions of the set  $\bar{t}$  into subsets  $\bar{t}_I, \bar{t}_{II}$  and the set  $\bar{x}$  into subsets  $\bar{x}_I, \bar{x}_{II}$  with  $\#\bar{t}_I = \#\bar{x}_I = n, n = 0, 1, \dots, a$ .

- The sum over partitions of  $\bar{s}$  and  $\bar{y}$ .

$$Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = \sum (-1)^n f(\bar{s}_{II}, \bar{t}) f(\bar{y}_I, \bar{x}) f(\bar{s}_I, \bar{s}_{II}) f(\bar{y}_{II}, \bar{y}_I) \\ \times K_n(\bar{y}_I|\bar{s}_I) K_{b-n}(\bar{y}_{II} + c|\bar{s}_{II}) K_{a+n}(\bar{s}_I, \bar{x}|\bar{y}_I + c, \bar{t}). \quad (3.9)$$

The sum is taken with respect to all partitions of the set  $\bar{s}$  into subsets  $\bar{s}_I, \bar{s}_{II}$  and the set  $\bar{y}$  into subsets  $\bar{y}_I, \bar{y}_{II}$  with  $\#\bar{s}_I = \#\bar{y}_I = n, n = 0, 1, \dots, b$ .

### 3.2 Integral representations

Now we give several representations for  $Z_{a,b}$  in terms of multiple contour integrals of Cauchy type. The formulas in terms of sums over partitions given above follow from the integral representations.

- $b$ -fold integrals.

$$Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = \frac{1}{(2\pi ic)^b b!} \oint_{\bar{w}} K_b(\bar{s} - c|\bar{z}) K_b(\bar{y}|\bar{z}) K_{a+b}(\bar{w}|\bar{t}, \bar{z} + c) f(\bar{z}, \bar{w}) \mathcal{F}_b(\bar{z}) d\bar{z}, \quad (3.10)$$

where

$$\mathcal{F}_b(\bar{z}) = \prod_{\substack{j,k=1 \\ j \neq k}}^b f^{-1}(z_j, z_k), \quad (3.11)$$

and  $d\bar{z} = dz_1, \dots, dz_b$ . We have used a subscript  $\bar{w}$  on the integral symbol in order to stress that the integration contour for every  $z_j$  surrounds the set  $\bar{w} = \{\bar{s}, \bar{x}\}$  in the

counterclockwise direction. We also assume that the integration contours do not contain any other singularities of the integrand. Similar prescription will be kept for all other integral representations considered below.

One more  $b$ -fold integral for  $Z_{a,b}$  has the form

$$Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) = \frac{(-1)^b f(\bar{y}, \bar{x}) f(\bar{s}, \bar{t})}{(2\pi i c)^b b!} \oint_{\bar{\eta}} K_b(\bar{z} | \bar{s}) K_b(\bar{z} | \bar{y} + c) K_{a+b}(\bar{x}, \bar{z} | \bar{\eta}) f(\bar{\eta}, \bar{z}) \mathcal{F}_b(\bar{z}) d\bar{z}. \quad (3.12)$$

Here  $\bar{\eta} = \{\bar{y}, \bar{t} - c\}$ .

- $a$ -fold integrals.

$$Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) = \frac{(-1)^a}{(2\pi i c)^a a!} \oint_{\bar{w}} K_a(\bar{z} | \bar{x} + c) K_a(\bar{z} | \bar{t}) K_{a+b}(\bar{y}, \bar{z} - c | \bar{w}) f(\bar{w}, \bar{z}) \mathcal{F}_a(\bar{z}) d\bar{z}, \quad (3.13)$$

where  $d\bar{z} = dz_1, \dots, dz_a$ . The integration contours surround the set  $\bar{w}$ , like in (3.10).

An analog of (3.12) has the form

$$Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) = \frac{f(\bar{y}, \bar{x}) f(\bar{s}, \bar{t})}{(2\pi i c)^a a!} \oint_{\bar{\eta}} K_a(\bar{t} - c | \bar{z}) K_a(\bar{x} | \bar{z}) K_{a+b}(\bar{\eta} - c | \bar{s}, \bar{z}) f(\bar{z}, \bar{\eta}) \mathcal{F}_a(\bar{z}) d\bar{z}. \quad (3.14)$$

The integration contours surround the set  $\bar{\eta} = \{\bar{y} + c, \bar{t}\}$ .

### 3.3 Recursions for $Z_{a,b}$

The partition function defined by Fig. 4 possesses several important properties. First, it is a symmetric function with respect to any set of variables  $\bar{y}$ ,  $\bar{x}$ ,  $\bar{s}$ , or  $\bar{t}$ . This property follows from the Yang–Baxter equation (2.5) (see e.g. [9, 10]).

The second property is that  $Z_{a,b}$  is a rational function decreasing at least as  $1/z$  at  $z \rightarrow \infty$ , where  $z$  is an arbitrary argument of the partition function. This property is almost evident. Consider an arbitrary horizontal (or vertical) line of the lattice. Note that  $a$ - and  $b$ -type vertices behave as 1 as  $z \rightarrow \infty$ , while the  $c$ -type vertex behaves as  $1/z$ . Thus, it is enough to show that at least one  $c$ -vertex is on the line. As there are different indices on the both sides of the line, moving along this line we must meet a  $c$ -type vertex somewhere. The corresponding statistical weight decreases at infinity.

The most important property of the partition function (or, what is the same, of the highest coefficient) is that the residues of  $Z_{a,b}$  in its poles can be expressed in terms of  $Z_{a-1,b}$  or  $Z_{a,b-1}$ . Since  $Z_{a,b}$  is a rational function in all its variables, this property formally allows us to fix the partition function unambiguously, provided we know  $Z_{a,b}$  for small  $a$  and  $b$ . It is easy to see that for  $a = 0$  or  $b = 0$   $Z_{a,b}$  coincides with  $K_n$ :

$$Z_{a,0}(\bar{t}; \bar{x} | \emptyset; \emptyset) = K_a(\bar{x} | \bar{t}), \quad Z_{0,b}(\emptyset; \emptyset | \bar{s}; \bar{y}) = K_b(\bar{y} | \bar{s}). \quad (3.15)$$

Thus, if we find the recursions of  $Z_{a,b}$  in its poles, we will fix it completely.

Consider, for example,  $Z_{a,b}$  as a function of  $s_b$  with all other variables fixed. Then it has simple poles at  $s_b = y_m$ ,  $m = 1, \dots, b$  and  $s_b = t_\ell$ ,  $\ell = 1, \dots, a$ . Due to the symmetry of  $Z_{a,b}$  over  $\bar{y}$  and over  $\bar{t}$  it is enough to find the residues at  $s_b = y_b$  and  $s_b = t_a$ .

**Proposition 3.1.** *The residue of  $Z_{a,b}$  at  $s_b = y_b$  is expressed in terms of  $Z_{a,b-1}$ :*

$$\text{Res } Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) \Big|_{s_b=y_b} = -cf(y_b, \bar{s}_{\bar{b}})f(\bar{y}_{\bar{b}}, y_b)f(y_b, \bar{x})Z_{a,b-1}(\bar{t}; \bar{x} | \bar{s}_{\bar{b}}; \bar{y}_{\bar{b}}). \quad (3.16)$$

Recall that  $\bar{s}_{\bar{b}} = \bar{s} \setminus s_b$ ,  $\bar{y}_{\bar{b}} = \bar{y} \setminus y_b$ .

*The residue of  $Z_{a,b}$  at  $s_b = t_a$  is expressed in terms of  $Z_{a-1,b}$ :*

$$\text{Res } Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) \Big|_{s_b=t_a} = cf(\bar{s}_{\bar{b}}, t_a)f(t_a, \bar{t}_{\bar{a}}) \sum_{p=1}^a g(x_p, t_a)f(\bar{x}_{\bar{p}}, x_p)Z_{a-1,b}(\bar{t}_{\bar{a}}; \bar{x}_{\bar{p}} | \{\bar{s}_{\bar{b}}, x_p\}; \bar{y}_{\bar{b}}). \quad (3.17)$$

Here  $\bar{x}_{\bar{p}} = \bar{x} \setminus x_p$ .

The proof is given in section 6.

*Remark.* The highest coefficient  $Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y})$  also has poles at  $x_j = t_k$  and  $x_j = y_k$ . The residues in these poles satisfy recursions similar to (3.16), (3.17) (see e.g. [9]). We do not present these recursions explicitly, because we do not use them.

## 4 Proofs of the sum formulas for $Z_{a,b}$

In this section we prove representations (3.3) and (3.6) for  $Z_{a,b}$ .

We begin with equation (3.3). It follows from the properties of  $K_n$  that  $Z_{a,b}$  defined by (3.3) is a symmetric function with respect to every set of variables and goes to zero as one of its arguments goes to infinity. The initial conditions (3.15) obviously are valid. Hence, it is enough to prove that  $Z_{a,b}$  defined by (3.3) and considered as a function of  $s_b$  possesses the following properties:

- it has poles only at  $s_b = y_k$ ,  $k = 1, \dots, b$  and at  $s_b = t_j$ ,  $j = 1, \dots, a$ ;
- the residues in these poles satisfy the recursions established in the previous section.

First we find the poles of  $Z_{a,b}$  defined by (3.3).

Due to the product  $f(\bar{w}_I, \bar{w}_{II})$  every single term in (3.3) may have poles at  $w_j = w_k$ . It is clear, however, that these singularities cancel each other due to the sum over partitions. More precisely, the residue at each  $w_j = w_k$  will vanish, due to opposite contribution of the terms  $(w_j \in \bar{w}_I, w_k \in \bar{w}_{II})$  and  $(w_j \in \bar{w}_{II}, w_k \in \bar{w}_I)$ .

Other poles of  $Z_{a,b}$  should coincide with the poles of the three  $K_n$  terms entering (3.3). If we set  $w_I = \{\bar{s}_I, \bar{x}_I\}$  and  $\bar{w}_{II} = \{\bar{s}_{II}, \bar{x}_{II}\}$ , then due to (3.5) we have

$$K_b(\bar{s} - c | \bar{w}_I) = (-1)^{b+k} K_k(\bar{s}_{II} - c | \bar{x}_{II}). \quad (4.1)$$



This  $K_k$  function has pole at  $s_b = x_j + c$ , if  $s_b \in \bar{s}_\Pi$  and  $x_j \in \bar{x}_\Pi$ . However in this case the product  $f(\bar{w}_I, \bar{w}_\Pi)$  contains the factor  $f(x_j, s_b)$ , which vanishes at  $s_b = x_j + c$ . Hence,  $K_b(\bar{s} - c|\bar{w}_I)$  does not produce poles in the r.h.s. of (3.3).

Thus, we conclude that the poles of  $Z_{a,b}$  coincide with the poles of the two remaining  $K_a(\bar{w}_\Pi|\bar{t})$  and  $K_b(\bar{y}|\bar{w}_I)$ , which are just at the points  $s_b = y_k$ ,  $k = 1, \dots, b$  and  $s_b = t_j$ ,  $j = 1, \dots, a$ . It remains to check the recursions (3.16) and (3.17).

We start with the first one: let  $s_b \rightarrow y_b$ . The pole occurs if and only if  $s_b \in \bar{w}_I$ . Let  $\bar{w}_I = \{\bar{w}_I, s_b\}$ . Using (A.1) we find

$$\text{Res } K_b(\bar{y}|\bar{w}_I, s_b) \Big|_{s_b=y_b} = -cf(\bar{y}_b, y_b)f(y_b, \bar{w}_I)K_{b-1}(\bar{y}_b|\bar{w}_I). \quad (4.2)$$

Substituting this into (3.3) and using (A.4) we obtain

$$\begin{aligned} \text{Res } Z_{a,b} \Big|_{s_b=y_b} &= (-1)^b cf(\bar{y}_b, y_b) \sum' K_{b-1}(\bar{s}_b - c|\bar{w}_I) \\ &\quad \times K_{b-1}(\bar{y}_b|\bar{w}_I) K_a(\bar{w}_\Pi|\bar{y}) f(\bar{w}_I, \bar{w}_\Pi) f(y_b, \bar{w}_I) f(y_b, \bar{w}_\Pi), \end{aligned} \quad (4.3)$$

where  $\sum'$  means that the sum is taken over partitions of the set  $\bar{w} \setminus s_b$ . It remains to observe that

$$f(y_b, \bar{w}_I) f(y_b, \bar{w}_\Pi) = f(y_b, \bar{s}_b) f(y_b, \bar{x}), \quad (4.4)$$

independently on a specific partition. Hence, re-denoting  $\bar{w}_I$  by  $\bar{w}_I$  we obtain

$$\begin{aligned} \text{Res } Z_{a,b} \Big|_{s_b=y_b} &= (-1)^b cf(y_b, \bar{s}_b) f(\bar{y}_b, y_b) f(y_b, \bar{x}) \\ &\quad \times \sum K_{b-1}(\bar{s}_b - c|\bar{w}_I) K_{b-1}(\bar{y}_b|\bar{w}_I) K_a(\bar{w}_\Pi|\bar{t}) f(\bar{w}_I, \bar{w}_\Pi), \end{aligned} \quad (4.5)$$

where now  $\bar{w} = \{\bar{s}_b, \bar{x}\}$ . The sum over partitions evidently gives  $Z_{a,b-1}$  with  $s_b$  and  $y_b$  omitted. We arrive at (3.16).

Consider now the residue of  $Z_{a,b}$  at  $s_b \rightarrow t_a$ . The pole occurs if and only if  $s_b \in \bar{w}_\Pi$ . Let  $\bar{w}_\Pi = \{\bar{w}_{II}, s_b\}$ . Due to (A.1) we have

$$\text{Res } K_a(\bar{w}_{II}, s_b|\bar{y}) \Big|_{s_b=t_a} = cf(t_a, \bar{t}_a) f(\bar{w}_{II}, t_a) K_{a-1}(\bar{w}_{II}|\bar{t}_a). \quad (4.6)$$

Substituting this into (3.3) we find

$$\begin{aligned} \text{Res } Z_{a,b} \Big|_{s_b=t_a} &= (-1)^b cf(t_a, \bar{t}_a) \sum' K_b(\bar{s}_b - c, t_a - c|\bar{w}_I) \\ &\quad \times K_b(\bar{y}|\bar{w}_I) K_{a-1}(\bar{w}_{II}|\bar{t}_a) f(\bar{w}_I, \bar{w}_{II}) f(\bar{w}_{II}, t_a) f(\bar{w}_I, t_a), \end{aligned} \quad (4.7)$$

where  $\sum'$  again means that the sum is taken over partitions of the set  $\bar{w} \setminus s_b$ . Using

$$f(\bar{w}_{II}, t_a) f(\bar{w}_I, t_a) = f(\bar{s}_b, t_a) f(\bar{x}, t_a), \quad (4.8)$$

we obtain

$$\begin{aligned} \text{Res } Z_{a,b} \Big|_{s_b=t_a} &= (-1)^b cf(t_a, \bar{t}_a) f(\bar{s}_b, t_a) f(\bar{x}, t_a) \sum' K_b(\bar{s}_b - c, t_a - c|\bar{w}_I) \\ &\quad \times K_b(\bar{y}|\bar{w}_I) K_{a-1}(\bar{w}_{II}|\bar{t}_a) f(\bar{w}_I, \bar{w}_{II}). \end{aligned} \quad (4.9)$$

Observe that

$$f(\bar{x}, t_a) K_b(\bar{s}_b - c, t_a - c | \bar{w}_1) \rightarrow 0, \quad \text{at } t_a \rightarrow \infty, \quad (4.10)$$

and this combination as a function of  $t_a$  has poles only at  $t_a = x_p$ ,  $p = 1, \dots, a$ . Hence, developing it over these poles we have

$$f(\bar{x}, t_a) K_b(\bar{s}_b, t_a - c | \bar{w}_1) = \sum_{p=1}^a g(x_p, t_a) f(\bar{x}_p, x_p) K_b(\bar{s}_b - c, x_p - c | \bar{w}_1). \quad (4.11)$$

Substituting this into (4.9) and re-denoting  $\bar{w}_{\text{ii}}$  by  $\bar{w}_{\text{II}}$  we arrive at

$$\begin{aligned} \text{Res } Z_{a,b} \Big|_{s_b=t_a} &= (-1)^b c f(t_a, \bar{t}_a) f(\bar{s}_b, t_a) \sum_{p=1}^a g(x_p, t_a) f(\bar{x}_p, x_p) \\ &\quad \times \sum K_b(\bar{s}_b - c, x_p - c | \bar{w}_1) K_b(\bar{y} | \bar{w}_1) K_{a-1}(\bar{w}_{\text{II}} | \bar{t}_a) f(\bar{w}_1, \bar{w}_{\text{II}}), \end{aligned} \quad (4.12)$$

where now  $\bar{w} = \{\bar{s}_b, \bar{x}\}$ . We see that the sum over the partitions gives exactly the highest coefficient  $Z_{a-1,b}(\bar{t}_a; \bar{x}_p | \{\bar{s}_b; x_p\}, \bar{y})$  and we reproduce (3.17). This ends the proof of relation (3.3).

Consider now  $Z_{a,b}$  defined by equation (3.6). The recursion in the pole at  $s_b = y_b$  for this representation can be checked in a manner similar to the one described above. The proof of the recursion in the pole at  $s_b = t_a$  is slightly different. This pole is in the product  $f(\bar{s}, \bar{t})$  in the formula (3.6). We have

$$\begin{aligned} \text{Res } Z_{a,b} \Big|_{s_b=t_a} &= c f(\bar{s}_b, t_a) f(t_a, \bar{t}_a) f(\bar{s}_b, \bar{t}_a) f(\bar{y}, \bar{x}) \\ &\quad \times (-1)^a \sum K_a(\bar{t} - c | \bar{\eta}_1) K_a(\bar{x} | \bar{\eta}_1) K_b(\bar{\eta}_{\text{II}} - c | \bar{s}_b, t_a) f(\bar{\eta}_1, \bar{\eta}_{\text{II}}). \end{aligned} \quad (4.13)$$

Consider the second line of (4.13) as a rational function of  $t_a$ . This rational function evidently vanishes as  $t_a \rightarrow \infty$ . Let us find the poles of this function. Suppose that  $t_a \in \bar{\eta}_{\text{II}}$ . Then due to (A.4)  $t_a$  disappears from the arguments of  $K_b(\bar{\eta}_{\text{II}} - c | \bar{s}_b, t_a)$ , but it remains in  $K_a(\bar{t} - c | \bar{\eta}_1)$ . There might be poles if  $t_a - c$  coincides with some element belonging to  $\eta_1$ , but they are compensated by the zeros of the product  $f(\bar{\eta}_1, \bar{\eta}_{\text{II}})$ . Hence, the rational function has no poles in this case.

Let now  $t_a \in \bar{\eta}_1$ . Then  $t_a$  disappears from  $K_a(\bar{t} - c | \bar{\eta}_1)$ , but it appears in  $K_a(\bar{x} | \bar{\eta}_1)$ , where we obtain the poles at  $t_a = x_p$ ,  $p = 1, \dots, a$ . Introducing  $\bar{\eta}_1 = \bar{\eta}_1 \setminus t_a$  and using (A.2), (A.4), we develop  $K_a(\bar{x} | \bar{\eta}_1)$  over the poles at  $t_a = x_p$  to rewrite (4.13) as

$$\begin{aligned} \text{Res } Z_{a,b} \Big|_{s_b=t_a} &= (-1)^{a-1} c f(\bar{s}_b, t_a) f(t_a, \bar{t}_a) f(\bar{s}_b, \bar{t}_a) f(\bar{y}, \bar{x}) \sum' \sum_{p=1}^a K_{a-1}(\bar{t}_a - c | \bar{\eta}_1) \\ &\quad \times g(x_p, t_a) f(\bar{x}_p, x_p) f(x_p, \bar{\eta}_1) K_{a-1}(\bar{x}_p | \bar{\eta}_1) K_b(\bar{\eta}_{\text{II}} - c | \bar{s}_b, x_p) f(\bar{\eta}_1, \bar{\eta}_{\text{II}}) f(x_p, \bar{\eta}_{\text{II}}), \end{aligned} \quad (4.14)$$

where  $\sum'$  means that the sum is taken over the partitions of the set  $\bar{\eta} \setminus t_a$  into the subsets  $\bar{\eta}_1$  and  $\bar{\eta}_{\text{II}}$ . Using

$$f(x_p, \bar{\eta}_1) f(x_p, \bar{\eta}_{\text{II}}) = f(x_p, \bar{t}_a) f(x_p, \bar{y} + c) = f(x_p, \bar{t}_a) f^{-1}(\bar{y}, x_p), \quad (4.15)$$

and re-denoting  $\bar{\eta}_i = \bar{\eta}_I$  we re-write (4.14) in the form

$$\begin{aligned} \text{Res } Z_{a,b} \Big|_{s_b=t_a} &= cf(\bar{s}_b, t_a) f(t_a, \bar{t}_a) \sum_{p=1}^a g(x_p, t_a) f(\bar{x}_p, x_p) \\ &\times f(\bar{s}_b, \bar{t}_a) f(x_p, \bar{t}_a) f(\bar{y}, x_p) (-1)^{a-1} \sum K_{a-1}(\bar{t}_a - c|\bar{\eta}_I) K_{a-1}(\bar{x}_p|\bar{\eta}_I) K_b(\bar{\eta}_I - c|\bar{s}_b, x_p) f(\bar{\eta}_I, \bar{\eta}_I). \end{aligned} \quad (4.16)$$

Looking at the expression in the second line of (4.16) one can easily recognize the highest coefficient  $Z_{a-1,b}(\bar{t}_a; \bar{x}_p|\{\bar{s}_b, x_p\}; \bar{y})$  defined by (3.6).

One can also prove that the representations (3.8), (3.9) satisfy the recursions (3.16) and (3.17). We will use, however, another method based on the multiple integral representations for  $Z_{a,b}$ .

## 5 Proofs of the integral representations for $Z_{a,b}$

All the integral representations listed in subsection 3.2 can be proved in a similar manner. Namely, they can be reduced to the sums over partitions listed in Section 3.1.

Consider for example (3.13). The only poles of the integrand within the integration contours are the points  $z_j = w_k$ . Evaluating the integral by the residues in these poles we obtain

$$Z_{a,b} = \sum K_a(\bar{w}_I - c|\bar{x}) K_a(\bar{w}_I|\bar{t}) K_{a+b}(\bar{y}, \bar{w}_I - c|\bar{w}) f(\bar{w}_I, \bar{w}_I), \quad (5.1)$$

where the sum is taken over partitions of  $\bar{w}$  into subsets  $\bar{w}_I$  and  $\bar{w}_I$  with  $\#\bar{w}_I = b$  and  $\#\bar{w}_I = a$ . Due to (A.4) we have

$$K_{a+b}(\bar{y}, \bar{w}_I - c|\bar{w}) = (-1)^a K_b(\bar{y}|\bar{w}_I), \quad (5.2)$$

and we immediately arrive at (3.4).

Dealing with contour integrals of rational functions we always have a possibility to calculate them by the residues in the poles outside the original integration contour. Consider the integrand in (3.13) as a function of some  $z_j$ . It behaves as  $1/z_j^3$  at  $z_j \rightarrow \infty$ , hence, the residue at infinity vanishes. The poles outside the original integration contour are in the points  $z_j = t_k$  and  $z_j = x_k + c$  (the poles at  $z_j = w_k + c$  are compensated by the zeros of the product  $f(\bar{w}, \bar{z})$ ). Due to the factor  $\mathcal{F}_a(\bar{z})$  the integrand also has poles at  $z_j = z_k \pm c$  for  $k \neq j$ . These last poles do not contribute to the final result (see Appendix B), thus, we can move the original contour surrounding  $\bar{z} = \bar{w}$  to the points  $\bar{z} = \bar{t}$  and  $\bar{z} = \bar{x} + c$

$$Z_{a,b} = \frac{1}{(2\pi ic)^a a!} \oint_{\bar{\xi}} K_a(\bar{z}|\bar{x} + c) K_a(\bar{z}|\bar{t}) K_{a+b}(\bar{y}, \bar{z} - c|\bar{w}) f(\bar{w}, \bar{z}) \mathcal{F}_a(\bar{z}) d\bar{z}, \quad (5.3)$$

where we have combined the sets  $\bar{t}$  and  $\bar{x} + c$  into one set  $\bar{\xi}$ . Now we can compute this integral by the residues using (A.1). However, it is more convenient first to transform slightly the integrand. Namely, applying (A.5) for all three  $K_n$  in (5.3) we obtain

$$Z_{a,b} = \frac{(-1)^{a+b}}{(2\pi ic)^a a!} \oint_{\bar{\xi}} K_a(\bar{t} - c|\bar{z}) K_a(\bar{x}|\bar{z}) K_{a+b}(\bar{w}|\bar{y} + c, \bar{z}) f(\bar{y}, \bar{w}) f(\bar{z}, \bar{\xi}) \mathcal{F}_a(\bar{z}) d\bar{z}. \quad (5.4)$$

Now all the poles are explicitly combined in the factor  $f(\bar{z}, \bar{\xi})$ . Hence, the result of the integration gives the sum over partitions of  $\bar{\xi}$  into  $\bar{\xi}_I$  and  $\bar{\xi}_{II}$  with  $\#\bar{\xi}_I = \#\bar{\xi}_{II} = a$ :

$$Z_{a,b} = (-1)^{a+b} \sum K_a(\bar{t} - c|\bar{\xi}_I) K_a(\bar{x}|\bar{\xi}_I) K_{a+b}(\bar{w}|\bar{y} + c, \bar{\xi}_I) f(\bar{y}, \bar{w}) f(\bar{\xi}_I, \bar{\xi}_{II}). \quad (5.5)$$

It remains to set

$$\begin{aligned} \bar{\xi}_I &= \{\bar{t}_I, \bar{x}_{II} + c\}, & \#\bar{t}_I &= \#\bar{x}_I = n, & n &= 0, \dots, a, \\ \bar{\xi}_{II} &= \{\bar{t}_{II}, \bar{x}_I + c\}, \end{aligned} \quad (5.6)$$

and after simple algebra we arrive at (3.8). Thus, the two sum formulas (3.4) and (3.8) are different representations of the same integral (3.13). Since the equation (3.4) was already proved, we automatically obtain the proof of the equation (3.8).

Similarly one can check that direct evaluation of the integrals by the residues within the original contours in the representations (3.10), (3.12), and (3.14) give respectively the sum formulas (3.3), (3.7), and (3.6). The sum formula (3.9) follows, for example, from (3.10) after moving the contours in this representation.

## 6 Proof of the recursion formulas for $Z_{a,b}$

The recursion (3.16) was pointed out already in [9] (see also [10]). Therefore we give only a sketch of the proof for this relation, while detailing the proof for the recursion relation (3.17).

We start with relation (3.16). Due to the symmetry of the partition function with respect to each set of variables, we can assume that the parameter  $y_b$  corresponds to the extreme left vertical line, while the parameter  $s_b$  corresponds to the extreme lower horizontal line (see Fig. 4). Then the pole at  $s_b = y_b$  occurs if and only if the extreme South-West vertex is of  $c$ -type. It is easy to see that as soon as the  $c$ -type of the extreme South-West vertex is fixed, all the vertices along the left and lower boundaries can be restored unambiguously. Namely, the vertices corresponding to the variables  $(s_b, \bar{y}_b)$ ,  $(\bar{s}_b, y_b)$ , and  $(\bar{x}, y_b)$  are of  $a$ -type, while the vertices corresponding to the variables  $(s_b, \bar{t})$  are of  $b$ -type. The product of the corresponding statistical weights gives us the prefactor in (3.16). The remaining sub-lattice is  $Z_{a,b-1}$ , in which  $s_b$  and  $y_b$  are excluded. Thus, we arrive at (3.16).

The recursion (3.17) is slightly more sophisticated. Let, as before,  $s_b$  correspond to the lower horizontal line, while  $t_a$  corresponds to the right vertical line. Then the pole at  $s_b = t_a$  occurs if and only if the extreme South-East vertex is of  $c$ -type. Then we can restore all the vertices along the lower horizontal line and a part of vertices along the right vertical line. Namely, the vertices corresponding to the variables  $(s_b, \bar{y}_b)$  are of  $b$ -type, while the vertices corresponding to the variables  $(\bar{s}_b, t_a)$  and  $(s_b, \bar{t}_a)$  are of  $a$ -type. The product of the corresponding statistical weights is  $g(s_b, t_a) f(s_b, \bar{t}_a) f(\bar{s}_b, t_a)$ . Thus, we obtain

$$\text{Res } Z_{a,b} \Big|_{s_b=t_a} = c f(t_a, \bar{t}_a) f(\bar{s}_b, t_a) \cdot \tilde{Z}_{a,b}, \quad (6.1)$$

where  $\tilde{Z}_{a,b}$  is given as a new partition function shown on Fig. 5.

$$\tilde{Z}_{a,b}(\bar{t}; \bar{x} | \bar{s}_{\bar{b}}; \bar{y}) =$$

Figure 5: The partition function  $\tilde{Z}_{a,b}(\bar{t}; \bar{x} | \bar{s}_{\bar{b}}; \bar{y})$ . The sublattice is obtained from the original lattice by removing the vertices  $(s_b, \bar{y})$ ,  $(s_b, \bar{t})$  and  $(\bar{s}_{\bar{b}}, t_a)$ .

Consider the remaining partition function as a function of  $t_a$ :  $\tilde{Z}_{a,b} = \tilde{Z}_{a,b}(t_a)$ . This variable corresponds to the extreme right vertical line (the shorter one) of the lattice. The indices at the ends of this line are different, hence, a  $c$ -type vertex should be somewhere on this line. Then

$$\tilde{Z}_{a,b}(t_a) = \sum_{p=1}^a \Gamma_p g(x_p, t_a), \quad (6.2)$$

where  $\Gamma_p$  do not depend on  $t_a$ . Due to the symmetry of  $\tilde{Z}_{a,b}$  over  $\bar{x}$  it is enough to find only  $\Gamma_a$ . All other coefficients  $\Gamma_p$  can be obtained from  $\Gamma_a$  via the replacement  $x_a \leftrightarrow x_p$ . The contribution from the term  $p = a$  in (6.2) occurs if and only if the lowest vertex on the short line is of  $c$ -type. Then all the remaining vertices on the short line are of  $a$ -type, while the  $t_a$ -independent coefficient is equal to  $Z_{a-1,b}$ , where  $s_b$  is replaced by  $x_a$ . Thus, we find

$$\Gamma_a = Z_{a-1,b}(\bar{t}_{\bar{a}}, \bar{x}_{\bar{a}} | \{\bar{s}_{\bar{b}}, x_a\}, \bar{y}) f(\bar{x}_{\bar{a}}, x_a), \quad (6.3)$$

and hence,

$$\Gamma_p = Z_{a-1,b}(\bar{t}_{\bar{a}}, \bar{x}_{\bar{p}} | \{\bar{s}_{\bar{b}}, x_p\}, \bar{y}) f(\bar{x}_{\bar{p}}, x_p). \quad (6.4)$$

Substituting this into (6.2) and using (6.1) we prove the recursion (3.17).

Starting from  $Z_{0,b}$  and  $Z_{a,0}$  and using the recursions (3.16), (3.17) we can construct iteratively  $Z_{1,b}$  for  $b = 1, 2, \dots$ , then  $Z_{2,b}$  for  $b = 2, 3, \dots$  and so on.

## Conclusion

We have presented several different formulas for the highest coefficient  $Z_{a,b}$ . We hope that at least some of them will be useful for further applications.

There exists at least one more representation for  $Z_{a,b}$  that we did not mention in this paper. It is analogous to the equation (A.3) for  $K_n(\bar{x}|\bar{y})$ . Such type of representations naturally appeared in the series of papers devoted to the universal description of the nested Bethe vectors (see [13] and references therein). In this approach the Bethe vectors are expressed through the modes of the generating series in the ‘current’ realization of a quantum affine algebra or Yangian double. The rational function  $K_n(\bar{x}|\bar{y})$  serves as a kernel of the integral transform which relates Bethe vectors with product of currents of the Yangian double (in case of the models with  $SU(2)$ -symmetry). In [14], analogous kernel for the integral transform in the ‘current’ approach to the universal Bethe vectors was constructed for the model with  $U_q(\mathfrak{gl}_3)$ -symmetry. Since such kernel can be naturally associated with highest coefficient, the rational limit of the kernels found in [14] yields one more representation of  $Z_{a,b}$ . The corresponding proof will be published elsewhere.

Turning back to the representations considered in this paper we should note that all of them are given in terms of sums over partitions or in terms of multiple integrals. Of course, it would be better to have a representation for  $Z_{a,b}$  in terms of a single determinant, as it has been done for  $K_n$  in the  $\mathfrak{gl}_2$  case. However the structure of our formulas leads us to conjecture that such a single determinant representation hardly exists.

If our conjecture is correct, then a determinant formula for the scalar product of a generic Bethe vector with the transfer-matrix eigenvectors should not exist. Indeed, since  $Z_{a,b}$  is a particular case of such scalar product, the existence of a (single) determinant formula for  $Z_{a,b}$  is a prerequisite for the existence of a determinant formula for the scalar product. This negative result, however, does not mean that there is no determinant representations for some cases of scalar products involving generic Bethe vectors. Some of such particular cases were considered in [10]. In our forthcoming publication [11] we will present one more determinant formula, which allows to calculate some form factors of local operators of the quantum XXX  $SU(3)$ -invariant Heisenberg chain.

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## A Properties of $K_n$

$K_n$  is symmetric function of  $x_1, \dots, x_n$  and symmetric function of  $y_1, \dots, y_n$ . It behaves as  $1/x_n$  (resp.  $1/y_n$ ) as  $x_n \rightarrow \infty$  (resp.  $y_n \rightarrow \infty$ ) at other variables fixed. It has simple poles at  $x_j = y_k$ . The residues in these poles can be expressed in terms of  $K_{n-1}$ :

$$\begin{aligned} \text{Res } K_n(\bar{x}|\bar{y}) \Big|_{x_n=y_n} &= -cf(y_n, \bar{y}_n)f(\bar{x}_n, y_n) \cdot K_{n-1}(\bar{x}_n|\bar{y}_n), \\ \text{Res } K_n(\bar{x}|\bar{y}) \Big|_{y_n=x_n} &= cf(x_n, \bar{y}_n)f(\bar{x}_n, x_n) \cdot K_{n-1}(\bar{x}_n|\bar{y}_n). \end{aligned} \tag{A.1}$$

Using (A.1) we can develop  $K_n(\bar{x}|\bar{y})$  with respect to the poles at  $y_n = x_p$ ,  $p = 1, \dots, n$ :

$$K_n(\bar{x}|\bar{y}) = \sum_{p=1}^n g(x_p, y_n) f(x_p, \bar{y}_n) f(\bar{x}_p, x_p) K_{n-1}(\bar{x}_p|\bar{y}_n). \quad (\text{A.2})$$

The equation (A.2) expresses  $K_n$  in terms of  $K_{n-1}$ . Continuing this process we arrive at the following representation

$$K_n(\bar{x}|\bar{y}) = \text{Sym}_{\bar{x}} \prod_{j=1}^n g(x_j, y_j) \prod_{j>k}^n f(x_j, y_k) f(x_k, x_j), \quad (\text{A.3})$$

where  $\text{Sym}_{\bar{x}}$  means the symmetrization over  $\bar{x}$ .

One can also easily check that  $K_n$  possesses the following properties:

$$K_{n+1}(\bar{x}, z - c|\bar{y}, z) = K_{n+1}(\bar{x}, z|\bar{y}, z + c) = -K_n(\bar{x}|\bar{y}), \quad (\text{A.4})$$

and

$$K_n(\bar{x} - c|\bar{y}) = K_n(\bar{x}|\bar{y} + c) = (-1)^n f^{-1}(\bar{y}, \bar{x}) K_n(\bar{y}|\bar{x}). \quad (\text{A.5})$$

**Proposition A.1.** *Let  $x_2 = x_1 - c$ . Then  $K_n(\bar{x}|\bar{y})$  as a function of  $x_1$  is holomorphic in the points  $\bar{y}$ .*

*Proof.* If  $x_2 = x_1 - c$ , then the functions  $t(x_1, y_k)$  and  $t(x_2, y_k)$  have poles at  $x_1 = y_k$ ,  $k = 1, \dots, n$  (see (3.1)). However the prefactor  $h(\bar{x}, \bar{y})$  contains the product  $h(x_2, \bar{y}) = g^{-1}(x_1, \bar{y})$ , that compensates these poles.

Due to the symmetry of  $K_n$  over  $\bar{x}$  the same property holds if  $x_k = x_j - c$  for arbitrary  $j$  and  $k$ . Then  $K_n(\bar{x}|\bar{y})$  is a holomorphic function of  $x_j$  in  $\bar{y}$ .

## B Spurious poles

Consider the integral representation (3.13). The integrand is a symmetric function of the integration variables  $\bar{z}$ . Hence, replacing  $K_a(\bar{z}|\bar{t})$  via (A.3) we obtain

$$Z_{a,b} = \frac{(-1)^a}{(2\pi ic)^a} \oint_{\bar{w}} \prod_{j=1}^a g(z_j, t_j) \prod_{j>k}^a (f(z_j, t_k) f^{-1}(z_j, z_k)) \cdot K_a(\bar{z}|\bar{x} + c) K_{a+b}(\bar{y}, \bar{z} - c|\bar{w}) f(\bar{w}, \bar{z}) d\bar{z}. \quad (\text{B.1})$$

The original integration contours surround the points  $\bar{w} = \{\bar{s}, \bar{x}\}$ . Our goal is to move these contours to the points  $\bar{\xi} = \{\bar{t}, \bar{x} + c\}$ . We do it successively starting from the contour for  $z_1$ .

Consider the integrand in (B.1) as a function of  $z_1$ . It has poles at  $z_1 = t_1, x_1 + c, \dots, x_a + c$  and at  $z_1 = z_2 + c, \dots, z_a + c$  (recall that the poles at  $z_1 = w_m + c$  are compensated by the zeros of the product  $f(\bar{w}, \bar{z})$ ). Consider the residue at  $z_1 = z_p + c$ . Then

$$f(\bar{w}, z_1) f(\bar{w}, z_p) \Big|_{z_1=z_p+c} = f^{-1}(z_p, \bar{w}) f(\bar{w}, z_p) = \prod_{k=1}^{a+b} \frac{w_k - z_p + c}{w_k - z_p - c}. \quad (\text{B.2})$$

We see that the integrand as a function of  $z_p$  becomes holomorphic at  $z_p = w_k$ ,  $k = 1, \dots, a+b$ . Hence, the integral over  $z_p$  vanishes, because the integration contour for  $z_p$  still surrounds the points  $\bar{w}$ . We conclude that the integration contour for  $z_1$  can be moved to the points  $t_1$  and  $\bar{x} + c$  without any additional contribution from the poles at  $z_1 = z_p + c$  for  $p = 2, \dots, a$ . We arrive at<sup>1</sup>

$$Z_{a,b} = \frac{(-1)^{a-1}}{(2\pi ic)^a} \oint_{t_1, \bar{x}+c} dz_1 \oint_{\bar{w}} dz_2 \dots dz_a \prod_{j=1}^a g(z_j, t_j) \prod_{j>k}^a (f(z_j, t_k) f^{-1}(z_j, z_k)) \times K_a(\bar{z}|\bar{x}+c) K_{a+b}(\bar{y}, \bar{z}-c|\bar{w}) f(\bar{w}, \bar{z}). \quad (\text{B.3})$$

Now we move the integration contour for  $z_2$  to the points  $t_1, t_2$  and  $\bar{x} + c$ . Similarly to the case considered above we do not obtain contributions from the poles at  $z_2 = z_p + c$  for  $p = 3, \dots, a$ . One additional pole arises at  $z_2 = z_1 - c$ . However, it is easy to check that taking the residue in this pole we make the integral over  $z_1$  vanishing. Indeed

$$g(z_1, t_1) f(z_2, t_1) \Big|_{z_2=z_1-c} = g(z_1, t_1) f^{-1}(t_1, z_1) = \frac{c}{z_1 - t_1 - c}, \quad (\text{B.4})$$

thus, the pole at  $z_1 = t_1$  disappears. On the other hand due to Proposition A.1,  $K_a(\bar{z}|\bar{x}+c)$  at  $z_2 = z_1 - c$  becomes a holomorphic function of  $z_1$  in the points  $\bar{x} + c$ . Thus, all the integrand as a function of  $z_1$  has no poles within the integration contour and therefore the integral vanishes. We conclude that the contour for  $z_2$  can be moved to the points  $t_1, t_2$  and  $\bar{x} + c$  without any additional contribution.

The process obviously can be continued. It is clear that setting  $z_k = z_\ell + c$  for  $\ell = k+1, \dots, b$  or  $z_k = z_\ell - c$  for  $\ell = 1, \dots, k-1$  we always obtain a function of  $z_\ell$ , which is holomorphic in the domain of the integration. Hence, the integral over  $z_\ell$  vanishes. Thus, we arrive at

$$Z_{a,b} = \frac{1}{(2\pi ic)^a} \oint_{t_1, \bar{x}+c} dz_1 \oint_{t_1, t_2, \bar{x}+c} dz_2 \dots \oint_{\bar{t}, \bar{x}+c} dz_a \prod_{j=1}^a g(z_j, t_j) \prod_{j>k}^a (f(z_j, t_k) f^{-1}(z_j, z_k)) \times K_a(\bar{z}|\bar{x}+c) K_{a+b}(\bar{y}, \bar{z}-c|\bar{w}) f(\bar{w}, \bar{z}). \quad (\text{B.5})$$

Obviously the integration contours for every  $z_k$  can be extended to the contour surrounding all the points  $\bar{t}$ , since the integrand as a function of  $z_k$  is holomorphic in  $t_{k+1}, \dots, t_a$ . After that we symmetrize the integrand over all integration variables using (A.3) and we finally obtain (5.3).

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<sup>1</sup>The obtained contour around the points  $t_1$  and  $\bar{x} + c$  is oriented in the clockwise direction. Changing the orientation of this contour we obtain in (B.3) the sign  $(-1)^{a-1}$  instead of  $(-1)^a$  in (B.1).



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